Khovanov-Rozansky homology and a graphical calculus for tensor products

Ben Webster

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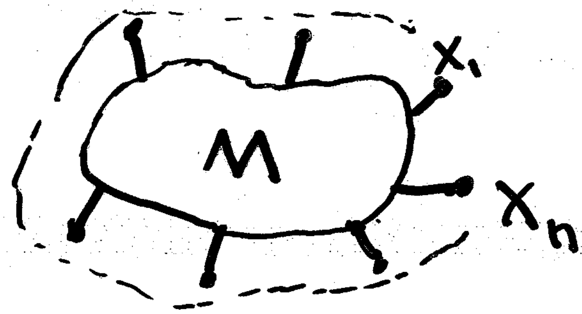
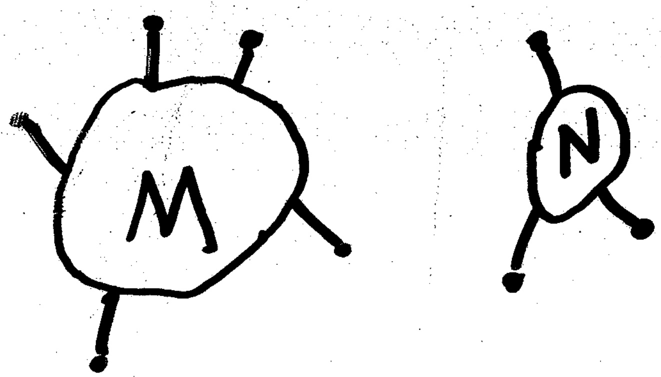
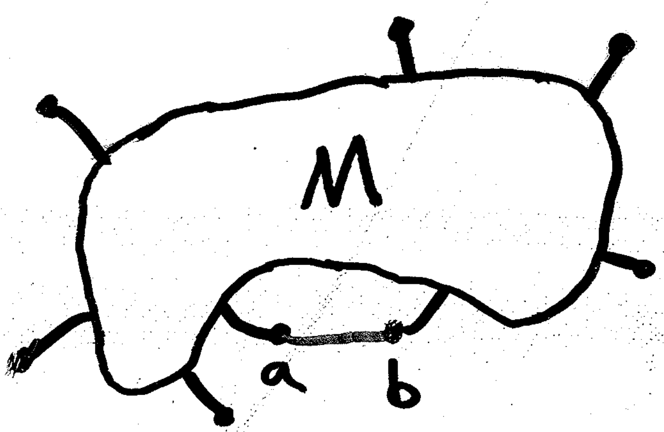
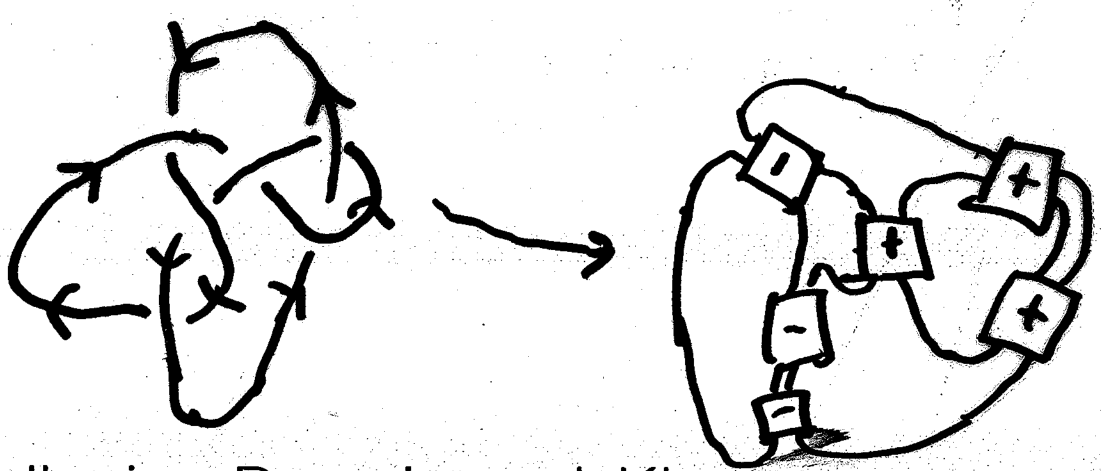
Graphical calculus.

	"M is a module over $k[x_1, \cdots x_n]$"
	$M \bigotimes_k N$
	$M \bigotimes_{k[x_a, x_b]} \frac{k[x_a, x_b]}{\langle x_a - x_b \rangle}$

To apply to knots, replace crossings with complexes:



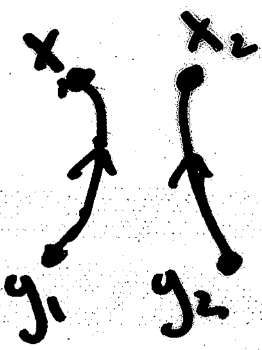
Following Rouquier and Khovanov, we make the substitution

 $0 \longrightarrow R(1) \xrightarrow{x_1 - x_2} B \longrightarrow 0$

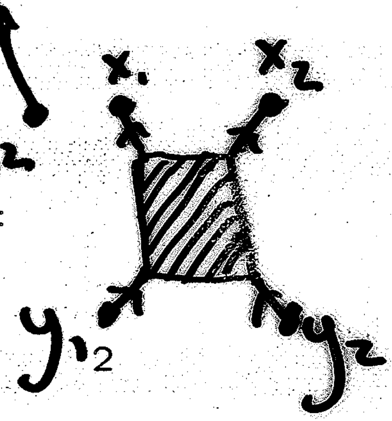
 $0 \longrightarrow B(-1) \xrightarrow{1} R(-1) \longrightarrow 0$

where

$$R = \frac{k[x_1, x_2, y_1, y_2]}{\langle x_1 - y_1, x_2 - y_2 \rangle} =$$



$$B = \frac{k[x_1, x_2, y_1, y_2]}{\langle x_1 - y_1 + x_2 - y_2, x_1x_2 - y_1y_2 \rangle} =$$



Enter the derived category.

This *should* be KR homology, but it's just the degree 0 part. What's wrong?

I should have used derived tensor product $\overset{L}{\otimes}$.

A brief introduction to $\overset{L}{\otimes}$:

$$A \overset{L}{\underset{R}{\bigotimes}} B = \cdots \to P_1 \underset{R}{\bigotimes} B \to P_0 \underset{R}{\bigotimes} B \in D^b(R-\text{mod})$$

where we have a free (projective) resolution

$$\cdots \to P_1 \to P_0 \to A$$

In particular $\overset{L}{\otimes}_k = \otimes_k$. So we need only change the final rule of our diagramatic calculus, by replacing one its factors with a free resolution.

$$\frac{k[x_a, x_b]}{\langle x_a - x_b \rangle} \approx k[x_a, x_b] \xrightarrow{x_a - x_b} k[x_a, x_b]$$

seems like a good choice.

Let $D^b(\mathcal{K})$ be the derived category of the homotopy category of complexes.

This a category of double complexes. We call the directions of our double complex $\boldsymbol{K}$ for "Koszul" and $\boldsymbol{C}$ for "cohomological." Two complexes are equivalent if there is a map inducing chain homotopy equivalence between K-homology.

If $M \in D^b(\mathcal{K})$ then $M' = M \overset{L}{\otimes}_{k[x_a,x_b]} \frac{k[x_a,x_b]}{\langle x_a - x_b \rangle}$ is the double complex with

$$(M')^{i,j} = M^{i,j} \oplus M^{i-1,j}$$

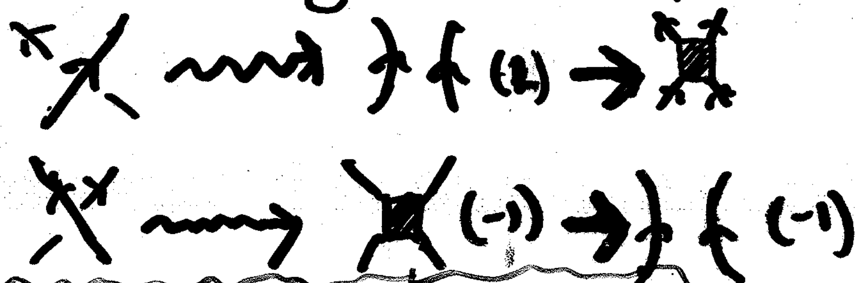
with differentials

$$\partial^K_{M'} = \begin{bmatrix} \partial^K_M & x_a - x_b \\ 0 & \partial^K_M \end{bmatrix}, \qquad \partial^C_{M'} = \begin{bmatrix} \partial^C_M & 0 \\ 0 & \partial^C_M \end{bmatrix}$$

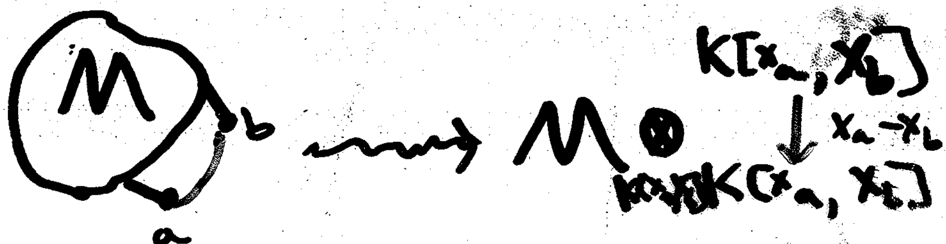
up to equivalence in $D^b(\mathcal{K})$.

Let's recap:

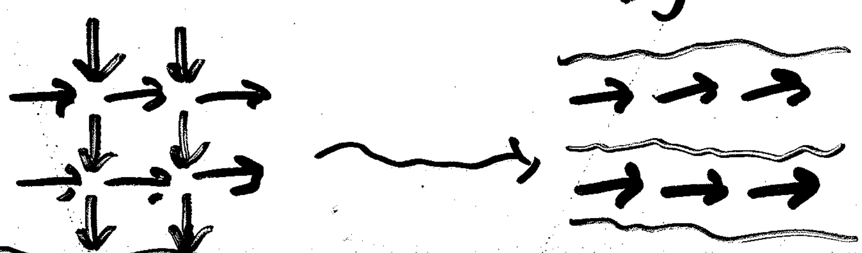
1. Crossings to complexes



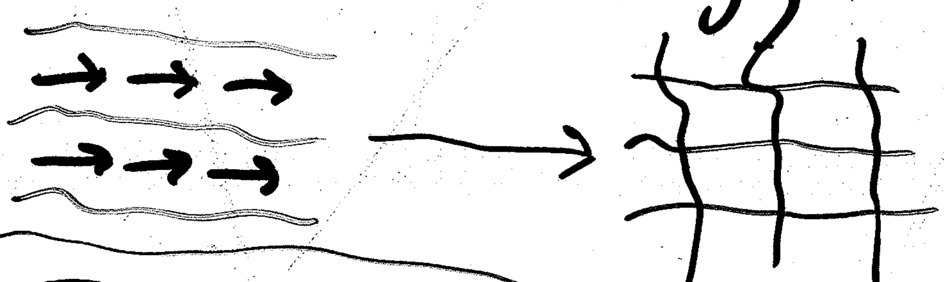
2. Connections to ⊗



3. Take K-homology



4. Take C-homology



5. Voila!

(remember:

C is horizontal →

K is vertical ↓)

(equivalence of double complexes)

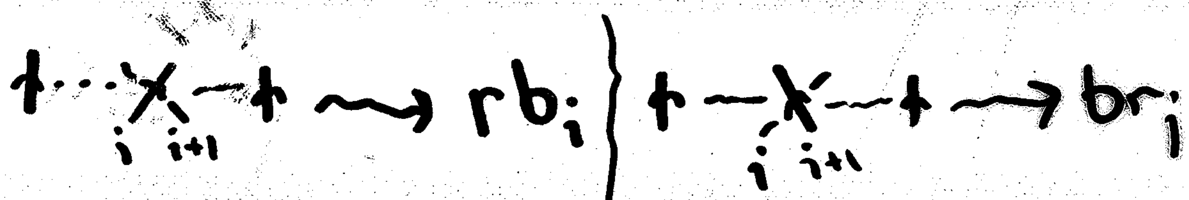
$M \xrightarrow{f} N$ s.t.

$H^i(M) \xrightarrow{H^i f} H^i(N)$

is a homotopy equivalence ∀ i

Comparing $\mathcal{KR}$ and HHH

Under our rules, the braid generators go to the right places, so the Koszul 0-homology of a braid σ agrees with Rouquier's complex $F(\sigma)$.



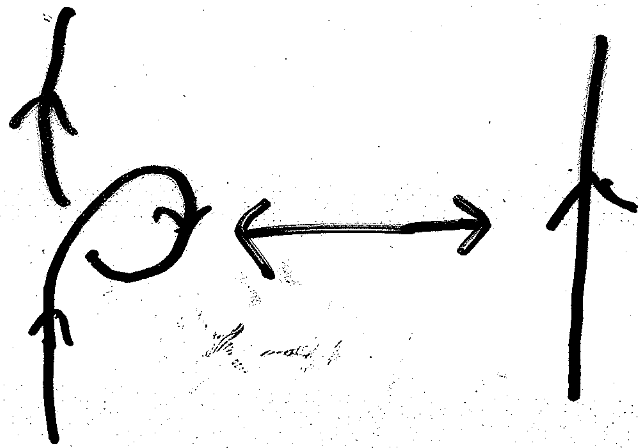
Proposition. *For σ a braid, $\mathcal{KR}(\sigma)$ is concentrated in K-degree 0, so $\mathcal{KR}(\sigma) \cong F(\sigma)$.*

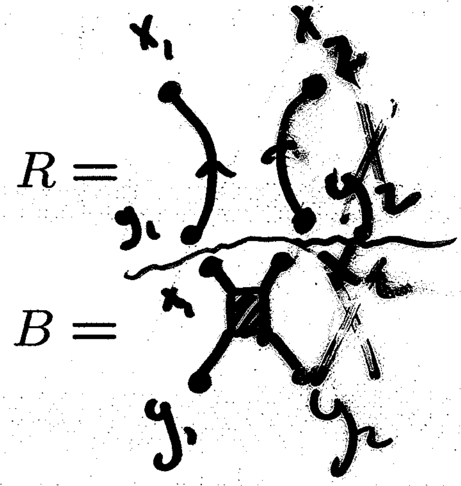
Thus, $\mathcal{KR}(\bar{\sigma})$ is $F(\sigma)\overset{L}{\otimes}_{R_{2n}} R_n$. Taking K-homology gives us $\mathrm{Tor}^i_{R_{2n}}(F^j(\sigma), R_n)$. Then taking C-homology gives us HHH.

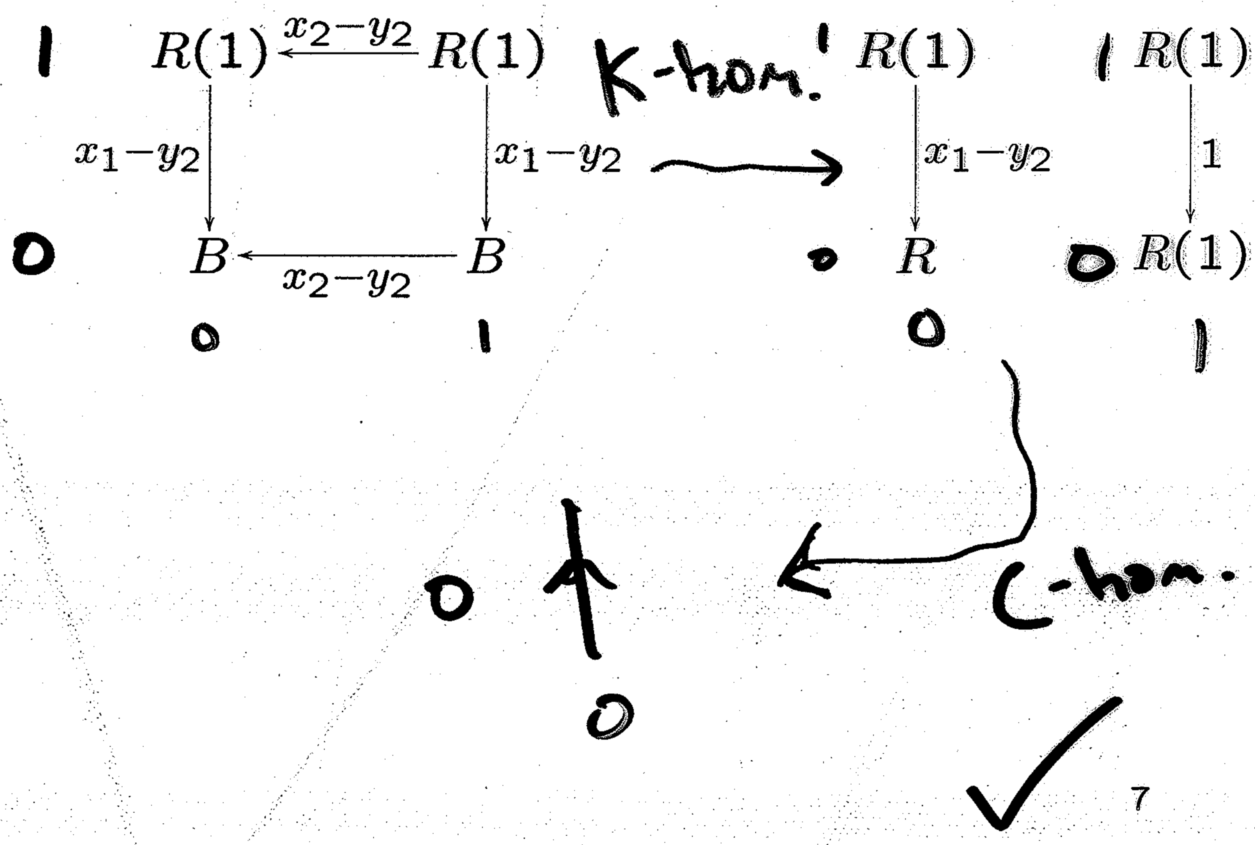
To show that we get the right answer for all diagrams, we only need to know Reidemeister invariance. Let's check those (quickly).

IIa and III are already done by Rouquier.

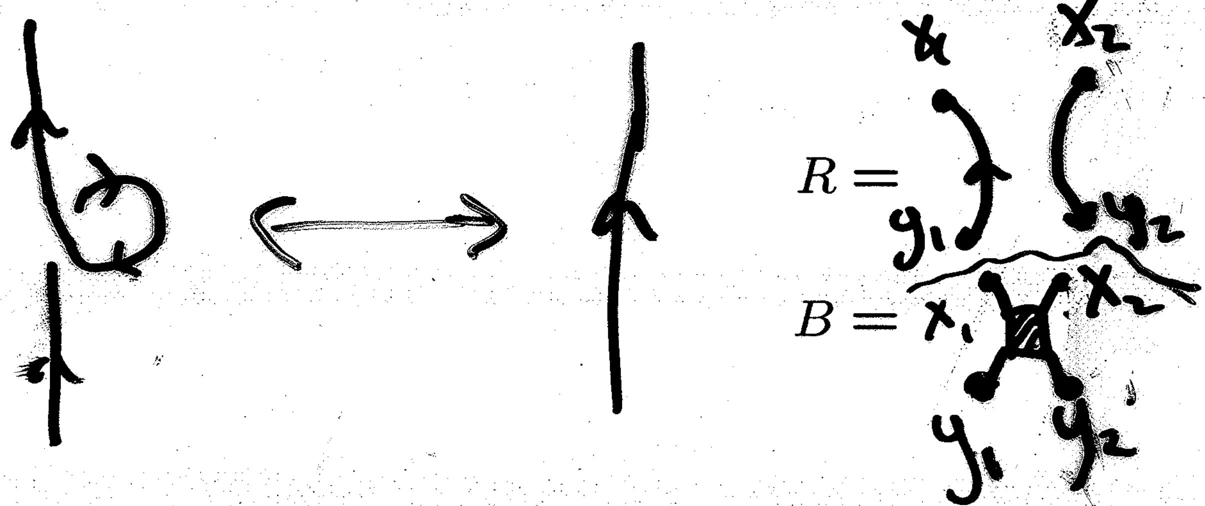
Invariance for Ia

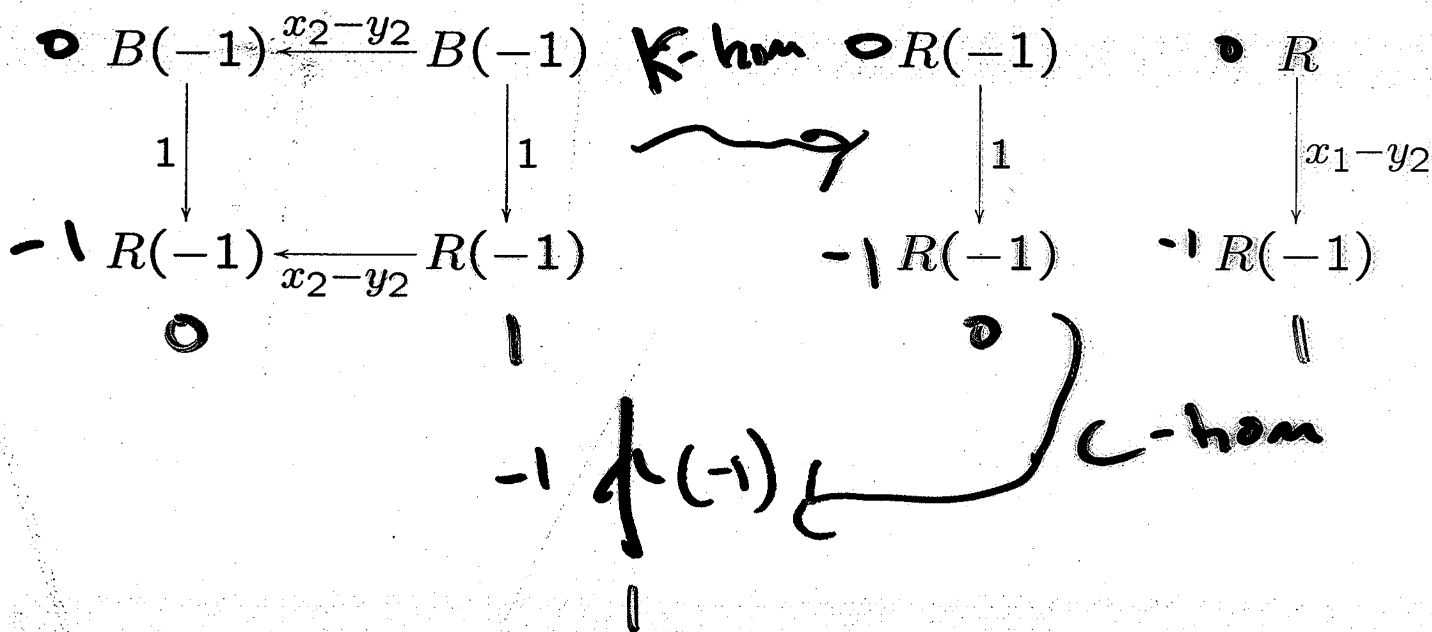






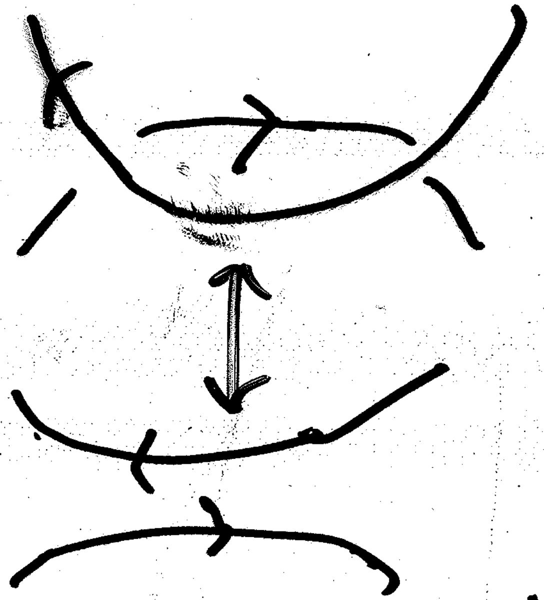
Invariance for Ib

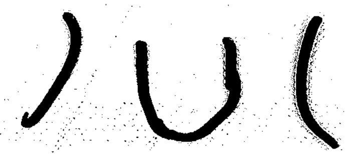




Unfortunately, we're not invariant under Ib. Thus, we have to add a grading shift to preserve invariance.

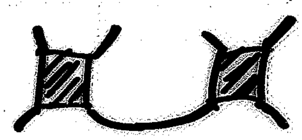
non-Invariance for IIb



$R_{1,2} =$ 

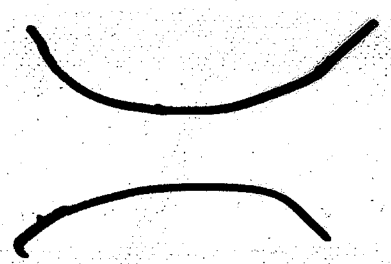
$B_1 =$ 

$B_2 =$ 

$B_{1,2} =$ 

$$\begin{array}{lccc}
1 & B_1 & \xleftarrow{w_1 - w_2} & B_1 \\
 & \downarrow{\scriptstyle 1 \oplus x_1 - z} & & \downarrow{\scriptstyle 1 \oplus x_1 - z} \\
0 & R_{1,2} \oplus B_{1,2}(-1) & \xleftarrow{w_1 - w_2} & R_{1,2} \oplus B_{1,2}(-1) \\
 & \downarrow{\scriptstyle (x_2 - z, 1)} & & \downarrow{\scriptstyle (x_2 - z, 1)} \\
-1 & B_2(-1) & \xleftarrow{w_1 - w_2} & B_2(-1) \\
 & 0 & & 1
\end{array}
\quad \xrightarrow{\text{K-hom}} \quad
\begin{array}{lclc}
1 & R_{1,2} & 1 & R_{1,2}(1) \\
 & \downarrow{\scriptstyle 1 \oplus x_1 - z} & & \downarrow{\scriptstyle x_1 - z \oplus 1} \\
0 & R_{1,2} \oplus B'(-1) & 0 & R_{1,2} \oplus R_{1,2}(1) \\
 & \downarrow{\scriptstyle (x_1 - z, f)} & & \downarrow{\scriptstyle (1, x_1 - z)} \\
-1 & R_{1,2}(-1) & -1 & R_{1,2} \\
 & 0 & & 1
\end{array}$$

c-hom

0 $\ker f \cong$ 

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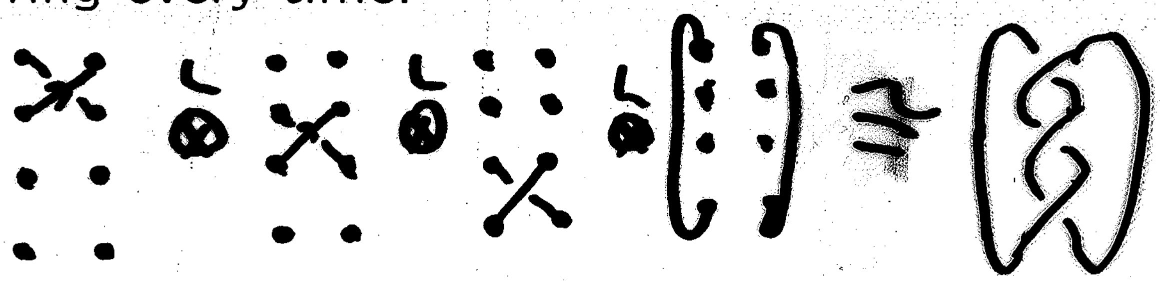
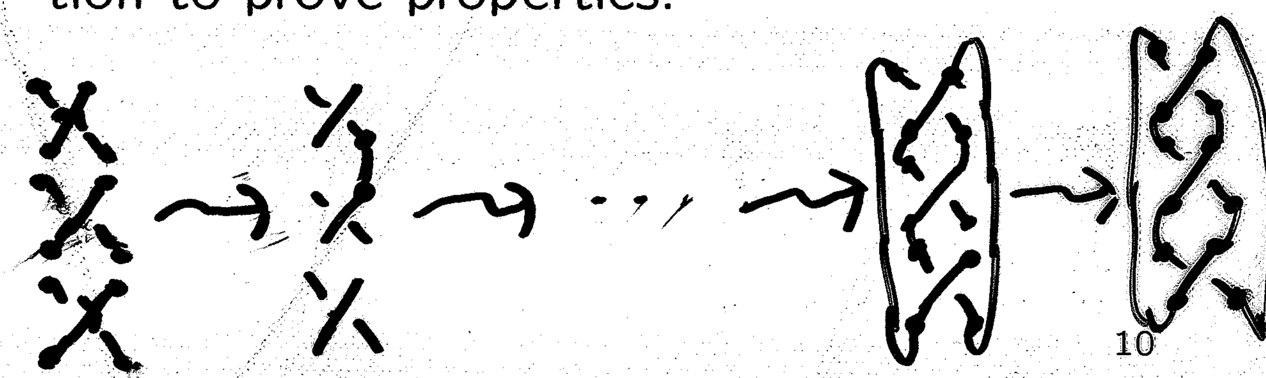
So, what does this have to do with actual computing? Flexibility.

There are many ways of computing $\mathcal{KR}$, each with it's own virtues. We can always choose the one that suits our purposes best.

Things are more convenient for computers if we add in all our variables at the beginning, so that we can take tensor product over the same ring every time.



But if we place all our crossings first, and then add the connections 1 by 1, we can use induction to prove properties.



The KR polynomial

To actually handle these, the easiest thing to do is take the Hilbert series

$$\widetilde{KR}_K(a,q,s) = \sum \left(H^{i,j}(\mathcal{KR}(K))\right)_k a^{2i} q^{2k} s^{2j}$$

This is a rational function, but if K is a knot, then

$$KR_K(a,q,s) = \frac{\widetilde{KR}_K(a,q,s)}{\widetilde{KR}_{0}(a,q,s)}$$

is a polynomial, and

$$KR_K(a,q,i) = H_K(a, q - q^{-1})$$

where $H_K(a,z)$ is the HOMFLYPT polynomial of K.

The skein relation comes from an exact triangle, as before.

Computations. Here's a list of our most recent computations of $KR(q,s,a)$.

0_1	1
3_1	$s^2a^4 - \frac{s^2a^2}{q^2} - \frac{q^2a^2}{s^2}$
4_1	$\frac{q^2}{s^2} - a^2s^2 - \frac{1}{a^2s^2} + 1 + \frac{s^2}{q^2}$
5_1	$-\frac{s^4a^6}{q^2} - q^2a^6 + \frac{s^4a^4}{q^4} + \frac{q^4a^4}{s^4} + a^4$
5_2	$-s^4a^6 + \frac{s^4a^4}{q^2} + q^2a^4 + s^2a^4 - \frac{s^2a^2}{q^2} - \frac{q^2a^2}{s^2} - a^2$
6_1	$a^4s^4 - \frac{a^2s^4}{q^2} - a^2s^2 + \frac{s^2}{q^2} - a^2q^2 + 2 + \frac{q^2}{s^2} - \frac{1}{a^2s^2}$
6_2	$\frac{s^4a^4}{q^2} + q^2a^4 + s^2a^4 - \frac{s^4a^2}{q^4} - \frac{s^2a^2}{q^2} - \frac{q^2a^2}{s^2} - \frac{q^4a^2}{s^4} - 2a^2 + \frac{1}{q^2} + \frac{q^2}{s^4}$
6_3	$\frac{q^4}{s^4} - a^2q^2 + \frac{q^2}{s^2} - \frac{q^2}{a^2s^4} - a^2s^2 - \frac{1}{a^2s^2} + 3 - \frac{a^2s^4}{q^2} + \frac{s^2}{q^2} - \frac{1}{a^2q^2} + \frac{s^4}{q^4}$
7_1	$\frac{s^6a^8}{q^4} + s^2a^8 + \frac{q^4a^8}{s^2} - \frac{s^6a^6}{q^6} - \frac{s^2a^6}{q^2} - \frac{q^2a^6}{s^2} - \frac{q^6a^6}{s^6}$
7_3	$\frac{q^4}{a^4s^4} - \frac{q^4}{a^6s^6} + \frac{q^2}{a^4s^2} - \frac{q^2}{a^6s^4} + \frac{q^2}{a^8s^6} - \frac{2}{a^6s^2}$ $+ \frac{1}{a^4} + \frac{s^2}{a^4q^2} + \frac{1}{a^8s^2q^2} - \frac{1}{a^6q^2} + \frac{s^4}{a^4q^4} - \frac{s^2}{a^6q^4}$
7_5	$\frac{s^6a^8}{q^2} + s^4a^8 + q^2s^2a^8 - \frac{s^6a^6}{q^4} - \frac{2s^4a^6}{q^2} - 2q^2a^6 - 2s^2a^6 - \frac{q^4a^6}{s^2}$ $+ \frac{s^4a^4}{q^4} + \frac{s^2a^4}{q^2} + \frac{q^2a^4}{s^2} + \frac{q^4a^4}{s^4} + 2a^4$
7_6	$-s^4a^6 + \frac{2s^4a^4}{q^2} + 2q^2a^4 + 2s^2a^4 - \frac{s^4a^2}{q^4} - \frac{2s^2a^2}{q^2} - \frac{2q^2a^2}{s^2}$ $- \frac{q^4a^2}{s^4} - 3a^2 + \frac{1}{q^2} + \frac{1}{s^2} + \frac{q^2}{s^4}$
7_7	$\frac{q^4}{s^4} - a^2q^2 + \frac{2q^2}{s^2} - \frac{2q^2}{a^2s^4} - 2a^2s^2 - \frac{2}{a^2s^2} + \frac{1}{a^4s^4} + 4 - \frac{a^2s^4}{q^2} + \frac{2s^2}{q^2} - \frac{2}{a^2q^2} + \frac{s^4}{q^4}$
8_{19}	$-\frac{q^6}{a^6s^6} + \frac{q^4}{a^8s^6} - \frac{q^2}{a^6s^2} + \frac{q^2}{a^8s^6} - \frac{1}{a^6s^2} + \frac{1}{a^8s^2} - \frac{1}{a^{10}s^6} - \frac{s^2}{a^6q^2} + \frac{1}{a^8s^2q^2} + \frac{s^2}{a^8q^4} - \frac{s^6}{a^6q^6}$
8_{20}	$\frac{a^4s^6}{q^2} - \frac{a^2s^6}{q^4} - 2a^2s^2 + a^4q^2s^2 + \frac{s^2}{q^2} + 1 - \frac{a^2q^4}{s^2} + \frac{q^2}{s^2}$
8_{21}	$-\frac{s^6a^6}{q^2} - s^4a^6 - q^2s^2a^6 + \frac{s^6a^4}{q^4} + \frac{s^4a^4}{q^2} + q^2a^4 + 3s^2a^4$ $+ \frac{q^4a^4}{s^2} - \frac{2s^2a^2}{q^2} - \frac{2q^2a^2}{s^2} - a^2$

(not shown: $8_2, 8_5, 8_7, 8_9, 8_{10}, 8_{16}$)

KR for alternating knots

The usual Khovanov homology of an alternating knot is determined by its signature and Jones polynomial. We hope there is an analogue for KR homology.

Conjecture. (Morrison, W.) *For all alternating knots,*

$$KR_K(a,q,s) = (is)^{-\sigma(K)} H_K(-isa, isq^{-1} + is^{-1}q).$$

(Note that this is a *much* nicer formula than that relating usual Khovanov homology to the Jones polynomial).

This is true for all alternating knots for which we have computed KR_K, but does not hold for 8_{19}, which is not alternating.