

A cobordism theory for \mathfrak{su}_3 knot homology.

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- Today's talk is on the convergence of these two approaches; a cobordism model for the \mathfrak{su}_3 theory, and hopefully a precursor to an \mathfrak{su}_n cobordism theory.

- 1 Quantum knot invariants
 - What is the \mathfrak{su}_3 spider?
 - What is a “planar algebra”?
 - Calculating knot invariants
 - A recipe for categorification
- 2 \mathfrak{su}_3 foams
 - Seamed cobordisms and local relations
 - What is a “canopolis”?
 - The \mathfrak{su}_3 homology knot invariant
 - Isomorphisms and decategorification
- 3 Computations
 - An algorithm for simplifying complexes
 - Reidemeister invariance

Part 1: Quantum knot invariants

- To begin, we'll recall how to calculate quantum knot invariants using “spiders”. For \mathfrak{su}_2 this is just another way of talking about the Kauffman bracket. The spider for \mathfrak{su}_3 was introduced by Kuperberg.
- In both cases, the spider is a type of *planar algebra*. It consists of $\mathbb{Z}[q, q^{-1}]$ -linear combinations of certain diagrams drawn in discs.

In the \mathfrak{su}_2 spider, the diagrams consist simply of unoriented arcs, with no crossings (like in the Temperley-Lieb algebra or the Kauffman skein module). There's just one relation:

$$\bigcirc = q + q^{-1}$$

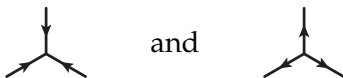
- In the \mathfrak{su}_3 spider, diagrams have oriented edges, and two types of vertices:



and



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- Now there are three relations:

$$\text{circle} = q^{-2} + 1 + q^2$$

$$\text{circle with arrows} = (q + q^{-1}) \text{ vertical line}$$

$$\text{square with arrows} = \text{two arcs} + \text{two arcs}$$

What is a “planar algebra”?

Definition

A ‘planar tangle’ is a diagram like this one:



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Planar tangles can be composed in obvious ways, by gluing smaller planar tangles into the internal discs of a larger planar tangle. Tangle composition is associative.

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- associates to a planar tangle T with internal discs $D_{k_1}, D_{k_2}, \dots, D_{k_n}$ and k_0 external marked points a map $\mathcal{P}_T : \mathcal{P}_{k_1} \times \mathcal{P}_{k_2} \cdots \times \mathcal{P}_{k_n} \rightarrow \mathcal{P}_{k_0}$,

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- and this association is compatible with tangle composition.

If \mathcal{C} is some monoidal category, a “planar algebra over \mathcal{C} ” associates an object of \mathcal{C} to each disc, instead of just a set, and associates an appropriate morphism in \mathcal{C} to each planar tangle.

Tangles and webs form planar algebras

- Tangles form a prototypical example of a planar algebra.
- Crossings and Reidemeister moves provide a presentation by generators and relations.

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Other examples...

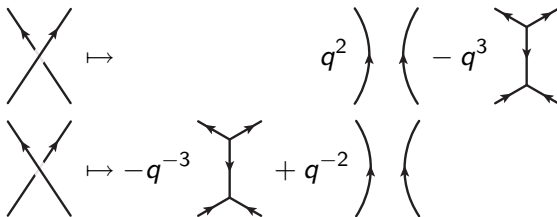
- Planar contraction of tensors.
- Subfactor planar algebras.

Calculating knot invariants

The \mathfrak{su}_3 quantum knot invariant is a map of planar algebras

Tangles \rightarrow **Spider** (\mathfrak{su}_3)

defined by



Example (Reidemeister I invariance)

$$\text{strand with loop} \mapsto q^2 \text{strand over circle} - q^3 \text{strand under circle}$$

Example (Reidemeister I invariance)

$$\begin{aligned}
 & \text{Diagram 1} \mapsto q^2 \text{Diagram 2} - q^3 \text{Diagram 3} \\
 & = q^2(q^{-2} + 1 + q^2) \text{Diagram 4} - q^3(q + q^{-1}) \text{Diagram 5}
 \end{aligned}$$

Example (Reidemeister I invariance)

$$\begin{aligned}
 & \text{Diagram of a strand with a loop} \mapsto q^2 \left\{ \text{Diagram of a strand with a small loop} \right\} - q^3 \left\{ \text{Diagram of a strand with a larger loop} \right\} \\
 & = q^2(q^{-2} + 1 + q^2) \left\{ \text{Diagram of a straight strand} \right\} - q^3(q + q^{-1}) \left\{ \text{Diagram of a straight strand} \right\} \\
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 \end{aligned}$$

A recipe for categorification

- Positive integers are secretly dimensions of vector spaces (and polynomials with positive integer coefficients are graded dimensions of graded vector spaces).

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- Positive integers are secretly dimensions of vector spaces.
- Integers are secretly Euler characteristics of complexes.
- This recipe tells us we should try to categorify quantum knot invariants by associating *complexes* to tangles, instead of linear combinations. We want to replace

$$\left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right) \mapsto q^2 \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) \left(-q^3 \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \right)$$

with some complex

$$\left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right) \mapsto \left(0 \longrightarrow q^2 \begin{array}{c} \diagup \\ \diagdown \end{array} \right) \left(\begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \xrightarrow{?} q^3 \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \longrightarrow 0 \right)$$

Part 2: What is a “seamed cobordism”?

Definition

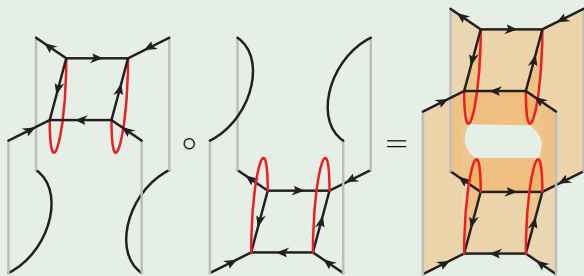
Given two \mathfrak{su}_3 webs, D_1 and D_2 , drawn in a disc D both with boundary points ∂ , a seamed cobordism from D_1 to D_2 is an oriented 2-complex F (the ‘foam’) embedded in a cylinder $D \times [0, 1]$, such that:

- The foam intersects the cylinder in $D_1 \times \{0\} \cup D_2 \times \{1\} \cup \partial \times [0, 1]$.
- Inside the cylinder, the foam is locally homeomorphic to either
 - an oriented disc
 - or three half-planes meeting at an edge.

Seamed cobordisms form a category, which we'll call **Foam** (\mathfrak{su}_3). The objects of this category are the \mathfrak{su}_3 web diagrams. We compose seamed cobordisms simply by stacking them one atop the other.

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Local relations

- Just as we had relations amongst \mathfrak{su}_3 webs, we need to introduce relations amongst \mathfrak{su}_3 foams.

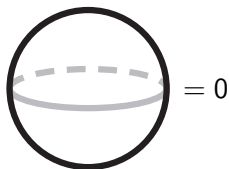
Local relations

- Just as we had relations amongst \mathfrak{su}_3 webs, we need to introduce relations amongst \mathfrak{su}_3 foams.
- Using these relations, we’ll discover certain isomorphisms amongst the objects in the \mathfrak{su}_3 foam category. These isomorphisms are ‘categorifications’ of the relations amongst \mathfrak{su}_3 webs.

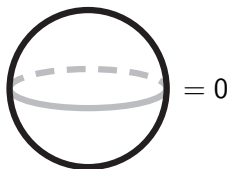
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- There are quite a few relations!

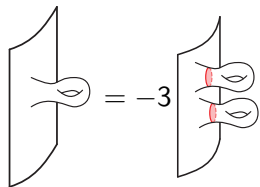
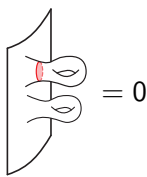
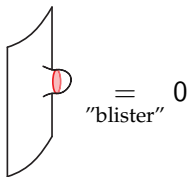
- “Closed foam” relations:



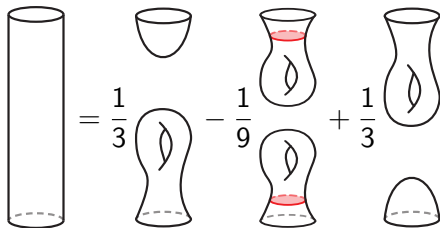
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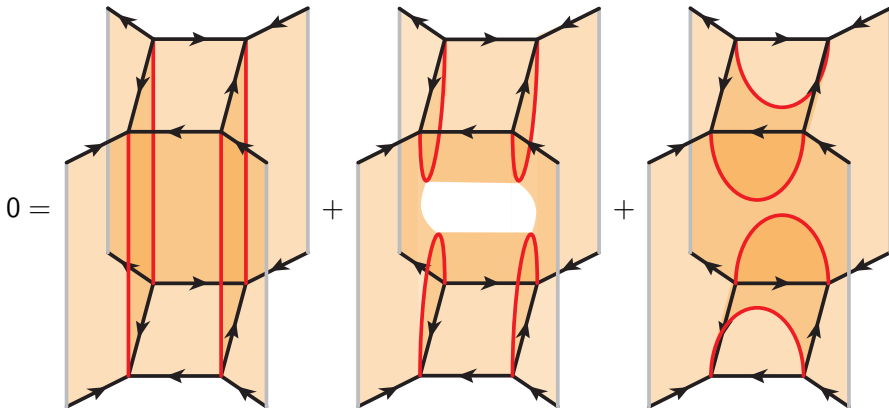
- "Sheet" relations:



- The “neck cutting” relation:



• The "three rocket" relation:



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The set \mathcal{C}_k associated to each disc D_k must be a *category*, and the map associated to a planar tangle must be *functorial*.

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- You can think of the objects of \mathcal{C}_k as living in the disc D_k , and the morphisms of \mathcal{C}_k as living in the ‘can’ $D_k \times [0, 1]$. You compose morphisms by stacking cans.

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- Functoriality says:

If you build a ‘city of cans’ using vertical and planar compositions, it doesn’t matter what order you do it in!

Example

The category of seamed cobordisms is actually a canopolis, with the objects forming the \mathfrak{su}_3 web planar algebra (just as sets; no linear combinations of diagrams, or relations).

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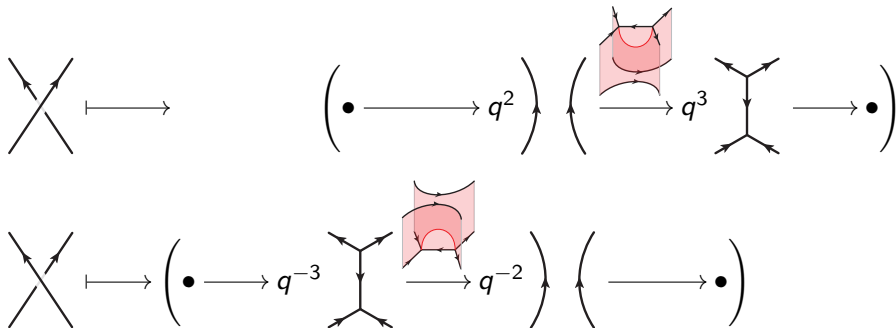
Example

Complexes in a canopolis form a planar algebra. [▶ how does this work?](#)

Finally we can say what the \mathfrak{su}_3 homology knot invariant is! It's a map of planar algebras, from tangles to up-to-homotopy complexes of foams,

$$\mathbf{Tangles} \rightarrow \mathbf{Kom}_{/htpy} \mathbf{Foam}(\mathfrak{su}_3)$$

defined by



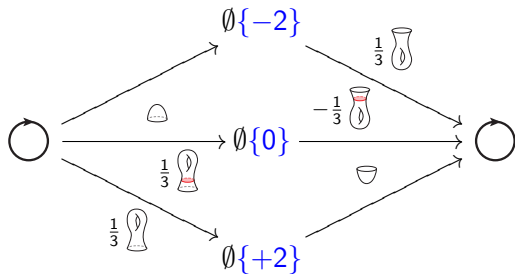
▶ What about tangle cobordisms?

Isomorphisms

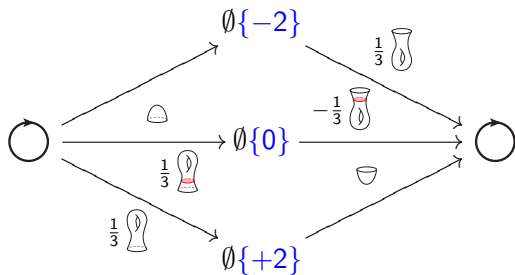
To begin, let's exhibit an isomorphism

$$\varphi : \bigcirc \xrightarrow{\cong} \emptyset\{-2\} \oplus \emptyset\{0\} \oplus \emptyset\{+2\}$$

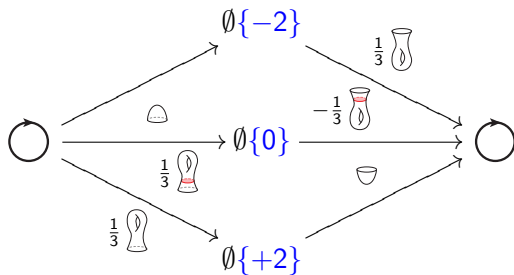
along with its inverse.



Now, let's check it really is an isomorphism!

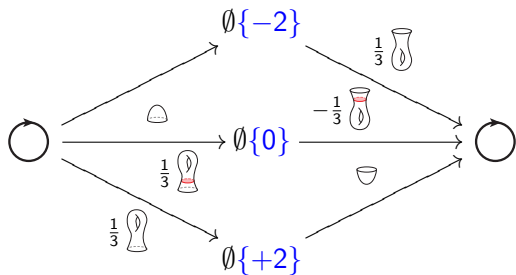


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$$\varphi^{-1}\varphi = \frac{1}{3} \begin{array}{c} \text{neck} \\ \text{cap} \end{array} - \frac{1}{9} \begin{array}{c} \text{neck with red band} \\ \text{neck with red band} \end{array} + \frac{1}{3} \begin{array}{c} \text{cup} \\ \text{neck} \end{array}$$

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$$\varphi^{-1}\varphi = \frac{1}{3} \begin{array}{c} \text{cup} \\ \text{neck} \end{array} - \frac{1}{9} \begin{array}{c} \text{cup} \\ \text{neck} \\ \text{cup} \end{array} + \frac{1}{3} \begin{array}{c} \text{cup} \\ \text{neck} \\ \text{cup} \end{array} = \text{neck cutting} = \text{cylinder} = \text{id} \circlearrowleft$$



And now the other way:

$$\varphi\varphi^{-1} = \begin{pmatrix} \text{cup} \\ -\frac{1}{3} \text{cup with seam} \\ \frac{1}{3} \text{cup} \end{pmatrix} \left(\frac{1}{3} \text{cup} \quad \frac{1}{3} \text{cup with seam} \quad \text{cup} \right)$$

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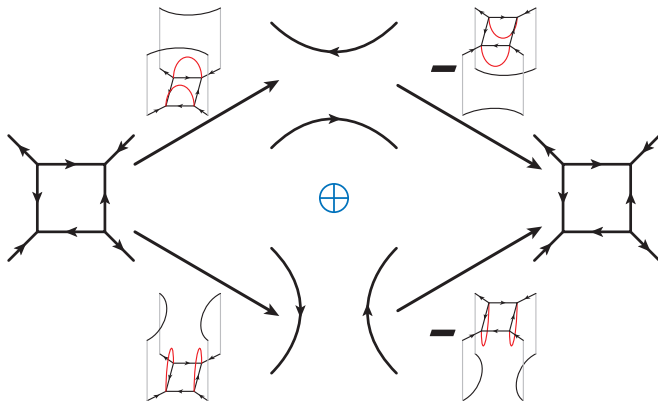
$$\begin{aligned} \varphi\varphi^{-1} &= \begin{pmatrix} \text{cup} \\ -\frac{1}{3} \text{cup with seam} \\ \frac{1}{3} \text{cup with seam} \end{pmatrix} \left(\frac{1}{3} \text{seam} \quad \frac{1}{3} \text{seam with seam} \quad \text{cup} \right) \\ &= \begin{pmatrix} \frac{1}{3} \text{seam} & & \\ -\frac{1}{9} \text{seam with seam} & -\frac{1}{9} \text{seam with seam} & -\frac{1}{3} \text{seam with seam} \\ \frac{1}{9} \text{seam} & \frac{1}{9} \text{seam} & \frac{1}{3} \text{seam} \end{pmatrix} \end{aligned}$$

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 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{id}_{\emptyset\{-2\} \oplus \emptyset\{0\} \oplus \emptyset\{+2\}}
 \end{aligned}$$



There's another isomorphism which 'explains' the rocket relation.



Decategorification

- We can “decategorify” a category, (not quite the usual way) obtaining an abelian group:

$$\mathcal{C} \rightsquigarrow \left\langle \text{Obj}(\mathcal{C}) \left| \begin{array}{l} A = B + C \text{ whenever} \\ A \cong B \oplus C \end{array} \right. \right\rangle$$

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- We can perform the same trick on a canopolis, obtaining a planar algebra of abelian groups.

Decategorifying **Foam** (\mathfrak{su}_3) to recover **Spider** (\mathfrak{su}_3)

- What is the planar algebra coming from the canopolis of \mathfrak{su}_3 foams?

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- We've seen enough to be sure it's some quotient of the planar algebra of \mathfrak{su}_3 webs:


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 \end{array} \right.$$

- Every relation we expect appears. But are they any isomorphisms we haven't noticed?

- We're still working on this! We can prove the corresponding result for \mathfrak{su}_2 using the fact that ‘gradings distinguish identities’ , but the same proof (surprisingly!) fails for \mathfrak{su}_3 .
- There's probably also an explanation in terms of \mathfrak{su}_3 representation theory: at generic values of q , there are no possible quotients.

Part 3: An algorithm for simplifying complexes

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- The algorithm presented here is an adaption of Bar-Natan's simplification algorithm for the \mathfrak{su}_2 theory.
- The algorithm has two steps:
 - 'decomposition'
 - 'cancellation'

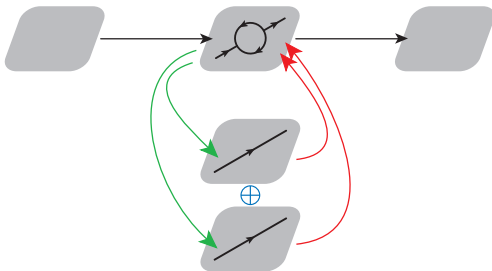


Decomposition Replace diagrams containing loops, bigons, or squares with their direct sum decompositions:

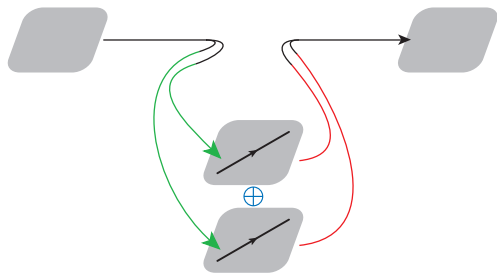
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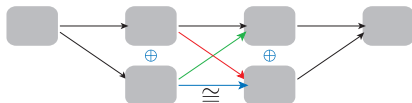
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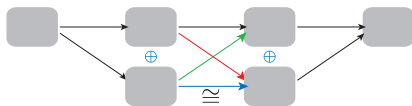
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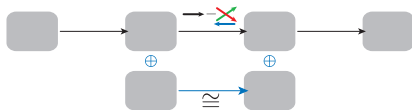
Cancellation Repeatedly cancel any isomorphisms appearing in the complex, using ‘Gaussian elimination’:



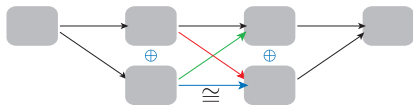
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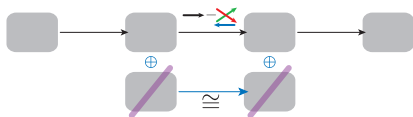
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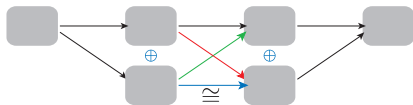


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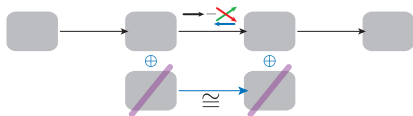


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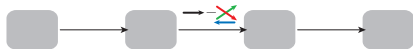
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which has a contractible direct summand, that can be homotoped away.



This algorithm makes choices along the way; we can cancel isomorphisms in different orders.

However, we can say a certain amount about the form of the result for knots and links (that is *closed* tangles):

Theorem

We eventually reach a complex in which only the empty diagram appears, because any closed non-empty \mathfrak{su}_3 web contains a loop, a bigon, or a square.

over \mathbb{Z} the final form is a complex of matrices with integer entries.

over \mathbb{Q} the final form is a complex with all morphisms being zero (because any non-zero matrix entry is invertible).

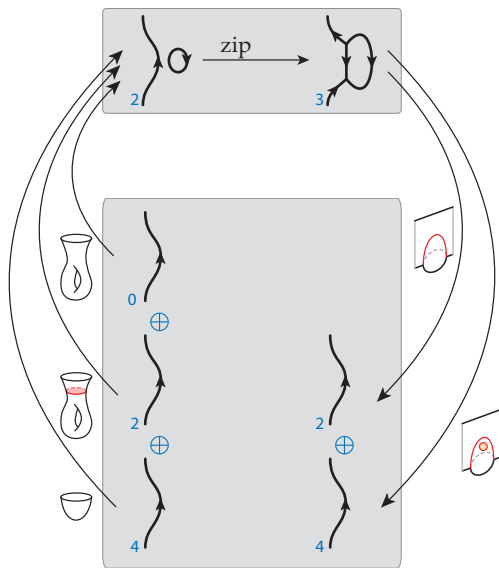
Reidemeister invariance

- Previous proofs of Reidemeister invariance required constructing explicit homotopies between complexes.

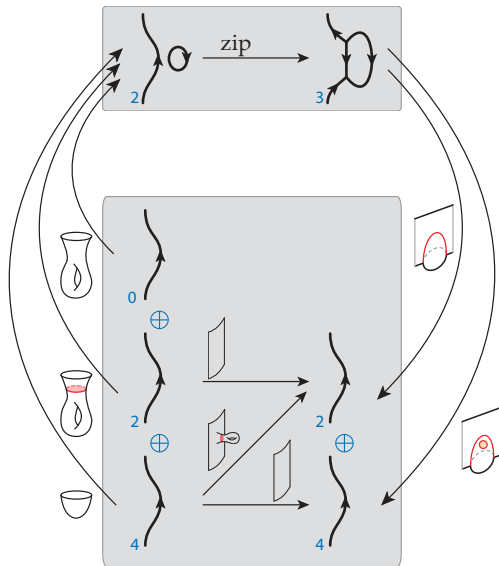
Reidemeister invariance

- Previous proofs of Reidemeister invariance required constructing explicit homotopies between complexes.
- The simplification algorithm ‘automates’ the process.
 - Simplifying the complexes associated to each side of a Reidemeister move always produces identical results.

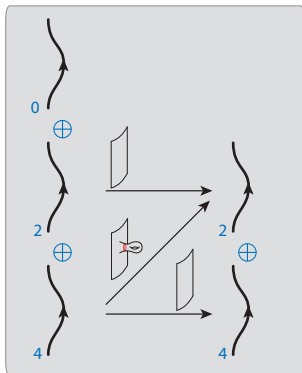
Let's schematically simplify the complex associated to a positive twist, $\mathcal{F} \left(\begin{array}{c} \text{loop} \\ \uparrow \end{array} \right)$:



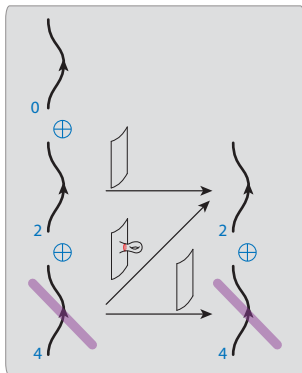
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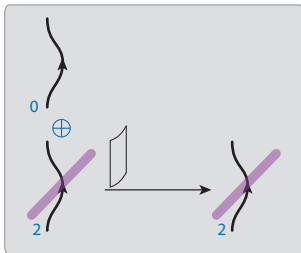
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The category of \mathfrak{su}_3 foams modulo local relations has several things going for it:

- It decategorifies to ‘the right thing’: the \mathfrak{su}_3 spider.
- It’s a suitable setting for a tangle homology theory.
- We know how to use the isomorphisms to simplify up-to-homotopy complexes.
- It’s ‘local’, meaning that we can do ‘divide and conquer’ calculations for knots, so it’s readily computable.

Appendix: link cobordisms

- Tangles form a planar algebra; in a natural way, tangle and link cobordisms (in \mathbb{R}^4) form a canopolis.
- Complexes in a canopolis form a planar algebra; complexes along with chain maps between them again form a canopolis.
- The \mathfrak{su}_3 homology invariant is actually functorial — it extends to a map between these canopolises.

◀ back to ' \mathfrak{su}_3 homology invariant'

Appendix: Grothendieck groups

- Usually, one “decategorifies” by taking the Grothendieck group:

$$\mathcal{G}(\mathcal{C}) = \left\langle \text{Obj}(\mathcal{C}) \left| \begin{array}{l} A = B + C \text{ whenever} \\ (0 \rightarrow B \rightarrow A \rightarrow C \rightarrow 0) \\ \text{is an exact sequence} \end{array} \right. \right\rangle$$


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- This doesn't work, as **Foam** (\mathfrak{su}_3) is not an abelian category; there are no notions of kernel, image or exactness.

Appendix: planar composition of complexes

- Given a quadratic tangle,  and a pair of complexes associated to the inner discs,


$$C_{\text{red}} = \left(\text{---} \rightarrow \text{---} \rightarrow \text{---} \rightarrow \text{---} \right)$$

$$C_{\text{blue}} = \left(\text{---} \rightarrow \text{---} \rightarrow \text{---} \rightarrow \text{---} \right)$$

The diagrams above represent sequences of four discs. In the red complex, the discs are connected by red arrows. In the blue complex, the discs are connected by blue arrows. Each disc contains a tangle of strands.

we need to define a new complex associated to the outer disc.

Appendix: planar composition of complexes

- Given a quadratic tangle,  and a pair of complexes associated to the inner discs,

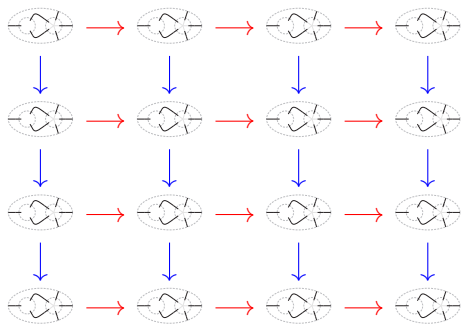
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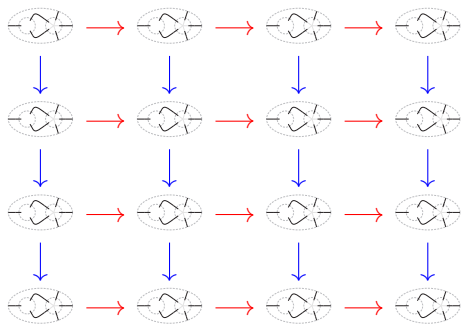
we need to define a new complex associated to the outer disc.

- We'll imitate the usual tensor product operation on complexes, making use of the planar tangle to combine objects and morphisms.

- First construct a double complex.



- First construct a double complex.



- Each horizontal arrow is the planar composition of an original red arrow with the identity on the right disc.
- Each vertical arrow is the planar composition of an original blue arrow with the identity on the left disc.

Appendix: gradings

- There's a grading on morphisms (essentially the Euler characteristic), additive under vertical and planar compositions.
- For \mathfrak{su}_2 , the grading distinguishes identities:

Theorem

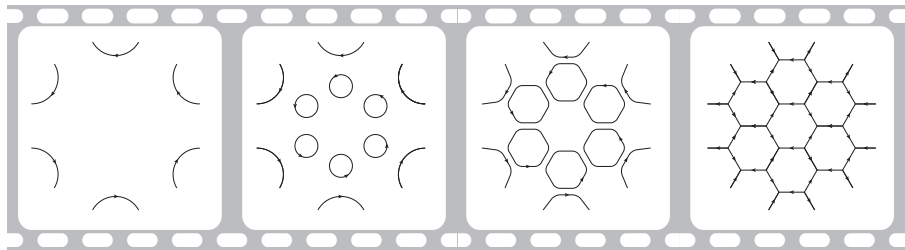
If D_1 and D_2 contain no loops, then

$$\mathrm{Hom}(D_1, D_2) = \bigoplus_{g \leq 0} \mathrm{Hom}_g(D_1, D_2)$$

and

$$\mathrm{Hom}_0 = \begin{cases} 0 & \text{if } D_1 \neq D_2 \\ \mathbb{Z}\langle \mathrm{id} \rangle & \text{if } D_1 = D_2. \end{cases}$$

- For \mathfrak{su}_3 , life is more complicated — there are nontrivial morphisms with non-negative gradings:



◀ back to 'Decategorification'